

The Mathematical Modeler

Tom Ho co-created the first interest-rate model that guaranteed consistency with the yield curve. His Ho-Lee model enabled the use of Black-Scholes technology for fixed-income derivatives. **By PETER CARR**

$$\pi h(T) + (1 - \pi)h^*(T) = 1$$

TOM HO WENT from the arcane mathematical field of Riemannian geometry to tackling the question of how to price a corporate bond.

Asked if that path involves a swerve somewhere, the president of New York-based financial engineering firm Thomas Ho Co. says he thinks of his career as linear. “It’s very much in one direction, as always my interest went in that direction,” Ho, 57, says. Maybe it takes a background in the study of curved space and higher dimensions first proposed by 19th-century German mathematician Bernhard Riemann to see it that way.

‘We mathematicians are humble; we only discover things of the beautiful world; we don’t invent beauty,’ Ho says.

Actually, you just have to look at it chronologically, Ho says. Born in Hong Kong, Ho went to the University of Warwick in Coventry, England, where he earned a bachelor’s degree in mathematics in 1973. He did well enough there that he was awarded a full scholarship to pursue a Ph.D. at the University of Pennsylvania in Philadelphia. At Penn,

Ho studied with mathematician Eugenio Calabi, whose Calabi-Yau manifold is one of the theoretical underpinnings of string theory in physics.

Ho’s dissertation was titled *The Spectrum of Grassmannians*. “Whatever that means!” Ho jokes, before explaining that it has to do with vibrations. As a drumhead vibrates a certain way, complex mathematical structures likewise oscillate in particular ways. “The original work done by Calabi was the vibration of an n -dimensional sphere in space and how many possible ways that a sphere can vibrate,” Ho says. “My thesis extended that to very abstract algebra where there is a lot of symmetry in this compact manifold, an object, and how many ways it can vibrate.”

That wasn’t enough to keep Ho occupied,

though. While many doctoral students were busy teaching, Ho’s scholarship allowed him time to take courses in other subjects. He began with physics and fluid dynamics, a natural topic because of his interest in how things flow in Riemannian space. From there, he became interested in econometrics and then macroeconomics. He took classes at Penn’s Wharton School of Business and studied finance with Irwin Friend, a professor who was then chair of Wharton’s finance department. One day, Friend confronted Ho. “Tom, what are you doing in my class?” Ho recalls Friend’s asking. When Ho replied that he was interested in extending his mathematical tools to finance, Friend offered Ho the office next to his.

AFTER GRADUATING in 1978 with a Ph.D. in math, Ho was hired by Friend to teach a course at Wharton. At the first class meeting, he was surprised to learn that his “students” consisted entirely of Wharton faculty. Finance at the time was undergoing a revolution. The swirl of ideas associated with the highly mathematical theory of option pricing created by Fischer Black and Nobel Memorial Prize in Economic Sciences laureates Myron Scholes and Robert Merton made continuing education a requirement for many professors. Ho asked the Wharton professors, many of whom were famous in their fields, to write their names on a seating chart.

After Wharton, Ho joined the finance department at New York University, where he began the research that led to the first arbitrage-free model of the term structure of interest rates, the Ho-Lee model. Ho became a full professor at NYU in 1985. A prolific author, Ho literally wrote the book on financial modeling with his longtime collaborator, Sang Bin Lee, a professor of finance at Hanyang



Ho began his research on interest rates as a young professor of finance.

University in Seoul. They're the authors of the 768-page *Oxford Guide to Financial Modeling* (Oxford University Press, 2003).

In 1987, Ho started his first company, Global Advanced Technology Corp., to develop applications of the Ho-Lee model. After selling the company to Berkeley, California-based financial software and consulting company Barra Inc., Ho took

a sabbatical in 1999. He and his wife spent 50 days riding bicycles across the U.S., from California to Florida. He started Thomas Ho Co. later that year.

When Ho joined NYU in 1978, he was one of nine assistant professors of finance hired from diverse fields such as math and operations research. They organized a seminar to educate themselves. For the group, Ho presented a mathematical 1977 working paper by John Cox, Jonathan Ingersoll and Stephen Ross. The 90-page paper laid out a theory for pricing interest-rate derivatives as a chapter in a more general theory of equilibrium asset pricing. The interest-rate theory, now known as

the Cox-Ingersoll-Ross model, was similar to other interest-rate models developed in the 1970s and early '80s. Derivatives are financial instruments whose value is derived from assets such as stocks or bonds.

The Cox-Ingersoll-Ross, Vasicek and Brennan-Schwartz models all focused on a theoretical construct called the short rate of interest as a starting point for their analysis. Like the Black-Scholes model used to price derivatives based on stocks, these interest-rate models used one or, at most, two so-called state variables to describe future states of the world. Though this simplistic approach made analysis of future uncertainty feasible, the interest-rate models had the drawback of treating the discount curve as something spit out by the model rather than as an input. The discount curve is the set of zero-coupon-bond prices of all possible maturities. Unfortunately, this model output didn't necessarily match market bond prices.

Presenting the CIR paper to his NYU colleagues jump-started Ho to do his own research on interest-rate derivatives. One of the main impediments to adapting the Black-Scholes theory to interest-rate

derivatives was that even though prices of different stocks can move independently of one another, the prices of bonds of different terms are related.

In particular, because money can be stored at no cost, a longer-dated zero-coupon bond cannot have a higher price than a nearer-dated one. As a special case of this observation, the price of a zero-coupon bond is pulled to par as it nears maturity. Short-rate models captured the pull to par, but they didn't necessarily respect no-arbitrage pricing relationships at other terms. The no-arbitrage condition requires no mispricing of assets relative to one another.

AS A FURTHER COMPLICATION, short-rate models required the modeler to specify the dynamics of something called the market price of short-rate risk. This unintuitive idea amounts to an insurance premium the market required to face the uncertainty of future interest rates.

With his training in n -dimensional Riemannian geometry, Ho took a different approach to interest-rate derivatives. Instead of modeling the dynamic evolution of one or two state variables, Ho set out to model the dynamics of an entire curve. The curve that Ho and Lee, then a graduate student at NYU, chose to describe as an input to their model was the discount curve, the output of the previous theories.

By lifting the valuation problem to this higher dimensional space, Ho and Lee were able to find a particularly simple model in which bond prices of all terms could randomly evolve without violating no-arbitrage constraints among themselves. By treating the discount curve as an input rather than an output, they could guarantee consistency with initially observed bond prices. And by modeling the prices of long-dated bonds instead of the yield of the nearest-dated bond, they avoided having to specify any notion of the market price of risk. As a result, Ho and Lee's model was the first to allow the application of equity-option-pricing technology to the rapidly expanding interest-rate-derivatives arena.

The example that Ho and Lee devised had particularly simple bond price dynamics. Over each discrete period, the discount function could either jump up or jump down by a fixed percentage relative to the expected future discount function. An up

Tom Ho

Ho-Lee model, key rate duration

EDUCATION

Received a bachelor's degree from the University of Warwick and a Ph.D. in mathematics from the University of Pennsylvania.

CAREER

Taught finance at New York University's Stern School of Business. Left academia in 1990 to run Global Advanced Technology, his first company. Was an executive vice president at Barra after it bought GAT in 1997. Now president of Thomas Ho Co.

PERSONAL

Age 57. Married, with one daughter.

jump followed by a down jump led to the same discount function as a down jump followed by an up jump.

LET'S TAKE A CLOSER LOOK at bond price dynamics in this binomial model. At time n and in state i , let $P_i^{(n)}(T)$ denote the price of a pure discount bond maturing in T periods. When $T = 0$, the bond is worth its face value, that is, $P_i^{(n)}(0) = 1$. As T becomes infinite, the bond price tends to zero $P_i^{(n)}(\infty) = 0$. At time n , the forward price of maturity 1 is $\frac{P_i^{(n)}(T+1)}{P_i^{(n)}(1)}$. If the term structure jumps up, each bond price jumps to a level that is higher than the forward price by a factor $h(T) > 1$:

$$P_{i+1}^{(n+1)}(T) = \frac{P_i^{(n)}(T+1)}{P_i^{(n)}(1)} h(T).$$

If the term structure jumps down, each bond price jumps to a level that is lower than the forward price by a factor $h^*(T) \in (0, 1)$:

$$P_i^{(n+1)}(T) = \frac{P_i^{(n)}(T+1)}{P_i^{(n)}(1)} h^*(T).$$

Notice that in this simple example, the perturbation functions h and h^* are independent of n and i . No arbitrage implies that there exists a number $\pi \in (0, 1)$, interpretable as a risk-neutral probability that the discount function realizes above the term structure of forward bond prices. As their binomial model is complete, π is unique, and the risk-neutral expected value of the bond price one period hence is the forward price, implying:

$$\pi h(T) + (1 - \pi) h^*(T) = 1.$$

Given a specification of the two perturbation functions, this risk-neutral probability is identified by solving this fundamental equation for π . As Ho and Lee further assumed path independence, they show that the only two possible perturbation functions are:

$$h(T) = \frac{1}{\pi + (1 - \pi)\delta^T}$$

and:

$$h^*(T) = \frac{\delta^T}{\pi + (1 - \pi)\delta^T},$$

where $\delta \in [0, 1]$. We see that this first example of a term structure model is fully described by the single parameter δ , which controls the ratio of $h^*(T)$ to $h(T)$. The closer δ gets to 1, the closer $h^*(T)$ gets to $h(T)$, and hence the lower is the bond price volatility.

Although they had developed this first example of

a term structure model by the early '80s, the two authors didn't immediately submit their discovery for publication. Using a pure-math-style approach to research, Ho wanted to find all of the possible arbitrage-free evolutions in their two-state discrete-time framework and to classify all of the different possibilities. "We mathematicians are humble; we only discover things of the beautiful world; we don't invent beauty," Ho says. "So to me, it was a very natural thing to think about a very simple Ho-Lee model, extend it in a discrete way to generalize it and have a classification theorem. This is the only way to do research." In 1984, the two authors entered their preliminary manuscript in an NYU working paper series. Practitioners who were grappling with the issues raised by earlier models snapped up the working paper, which was reprinted six times.

Once they had the general result, the authors submitted their paper to the *Journal of Finance*, the publication of the American Finance Association. During the refereeing process, though, Ho and Lee agreed to shorten the paper, removing the classification theorem. Nowadays, when people talk about the Ho-Lee model, they're referring to the simple example that makes up the paper as published in 1986. Of course, as soon as the paper came out, others set out to generalize it.

In the late '80s, Ho developed key rate duration, his second major contribution to finance. Key rate duration measures the sensitivity of a security or portfolio to small shifts in liquid interest rates at various terms. "The crux of understanding key rate duration is the Ho-Lee model because an arbitrage-free model says you can give me any shape of the yield curve; it still works," Ho says. Key rate duration in effect uses the model to hedge against its own deficiencies.

Taking model risk into account is an intrinsic part of financial engineering, Ho says, given that all models have limitations. "If you have a perfect model, you're in the real world already," he says. "It's not a model anymore." **B**

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Cheat Sheet

Type **PKRD <Go>** to calculate key rate durations. Type **OASN NEW <Go>** to calculate key rate duration for a selected mortgage security.