

# **A Multi-Factor Binomial Interest Rate Model with State Time Dependent Volatilities**

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## **ABSTRACT**

This paper presents a multifactor arbitrage-free interest rate model, based on a binomial lattice framework, that avoids negative and unreasonable high interest rate levels, and the model provides interest rates movement that is consistent with historical interest rate experience. The model is tested empirically using monthly swaption prices over a one year sample period. The results show that the model can be calibrated to the observed swaption prices quite accurately. The model supports the two-factor model over the one-factor model, and suggests that the implied yield curve movements from the swaption prices are more akin to a normal model than a lognormal model. This interest rate model is specified by a set of difference equations and is therefore simple to implement. The recombining lattice also provides accuracy in pricing a broad range of interest rate contingent claims. For these reasons, the model has many applications in securities valuation, risk management and regulatory compliance.

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## I . Introduction

This paper presents an interest rate model that has the following desirable attributes. The model is (1) arbitrage-free that fits the observed yield curve; (2) recombining in a binomial lattice framework; (3) multi-factor that can capture the changing shape of the yield curves. Lastly, (4) the model has multiple term structures of volatilities that are time and state dependent, such that the interest rates movements are mean reverting, exhibiting no negative or unreasonably high interest rates.

Attribute (1) ensures that the model fits the observed yield curve. By way of comparison, the Cox, Ingersoll and Ross (1977) and Vasicek (1977) models do not automatically fit the spot yield curve. Attribute (2) can value options and early exercise options, like American and Bermudan options, using the backward substitution approach. In contrast, the Heath, Jarrow and Morton approach (1989), Longstaff, Santa-Clara, Schwartz “String” model (2000) Brace, Gatarek, Musiela/Jamshidian model (1997) require numerical procedures and Monte-Carlo simulations to determine the valuation of most of these securities.

Attribute (3) captures much of the yield curve upward or downward sloping movements. Using Litterman and Scheinkman (1991) principal component method, a two-factor model and three-factor model can be shown to capture 94% and 97% of the historical yield curve movements (see Ho-Lee (2004a)) respectively. Therefore the proposed model, using a multi-factor binomial model with fewer than three factors can effectively capture the historical yield curve movements, something that Black, Derman Toy (1990); Black-Karasinski (1991), one factor models of Hull-White (1990), Ho-Lee (1984) cannot do.

Attribute (4) ensures that the model is consistent with the historical interest rate behavior. This model is consistent with Cheyette (1997) that the interest rate process tends to be lognormal when interest rates are low and the process becomes normal when interest rate levels are high. Intuitively, when interest rates are low, the standard deviation of the interest rate level must necessarily be small to avoid negative interest rates. However, when the interest rate level is high, the variation can no longer be proportional to the interest rate level to avoid excessively high interest rates. And the interest rates tend to exhibit mean reversion. Black, Derman and Toy (1989) does not have negative interest rates but the interest rates can be extremely high. Ho-Lee (1986, 2004) being a normal model has negative interest rates.

This paper also provides empirical tests of the model, based on 70 at-the-money swaption prices on each date, monthly data from June 30, 2003 to June 30, 2004. There are four main results. (1) The model can be calibrated to the swaption prices, using only six parameters, for each date with an average error ranging from 1% to 3%. The average errors vary with the dates. (2) The two-factor model provides significantly higher explanatory power than the one factor model, based on the Davidson and Mackinnon’s C test. (3) The calibrated interest rates movement in general behaves more like a normal model than a lognormal model. (4) The implied “instantaneous” principal movements of the yield curve are consistent with those estimated from the historical data, suggesting that the market prices do incorporate the appropriate correlations of the interest rates. The

simulated yield curve can be upward or downward sloping, and the movement exhibits mean reversion. All these behaviors are consistent with the historical experiences.

Our model is straightforward to implement. The model is specified by a set of difference equations, without any numerical estimation procedure as in the Black, Derman and Toy model and any tree pruning numerical methods as in the Hull and White model. The model is practical because it can be used for a broad range of securities, based on few intuitive assumptions. In combining all the attributes mentioned above, the model finds many applications. Beyond providing accuracy and speed in valuing a broad range of interest rate contingent claims, the model can be used for some of the recent challenges in financial modeling.

For example, International Financial Reporting Standard 39 regulation requires public firms to provide fair valuation of balance sheet items. Also, for financial disclosures, financial models are required to provide risk measures in economic value (and not book value) like Value-at-Risk. Sarbanes-Oxley regulations require transparency in business processes, including valuation of balance sheet items. Auditors and regulators prefer the interest rate models to have the properties beyond those typically used for valuation in trading or asset management. Our multi-factor binomial interest rate model with state and time dependent volatilities can address all these requirements.

The paper begins with Section B that provides the assumptions of the model, the specification of the model and the algorithm in valuing a bond. Section C presents the extensions of the model to multi-factor and the model of a T-period maturity discount bond. Section D reports the empirical results in calibrating the interest rate model to the swaption prices. Section E describes the behavior of the interest rates movement according to the interest rate model. Section F contains the implications of the model. Finally, section G contains the conclusions.

## **II. Descriptions of the Model**

The model is an extension of the Ho-Lee (1986) model. The Ho-Lee model assumes the forward volatility of the interest rate movement to be a constant, independent of state and time. However, the paper noted that the methodology can be extended to incorporate state and time dependent volatilities<sup>1</sup>. We here provide such a model.

### *A. Model Assumptions*

For clarity of exposition, we begin with a one-factor model, which we will generalize to multi-factor model in the following section. This is a discrete time model where all agents buy and sell securities without any market friction. We use a recombining binomial lattice model to specify the interest rate uncertainties, with the upstate and the

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<sup>1</sup> The paper noted in the footnote: "In general,  $h(T)$ ,  $h^*(T)$ , and  $\pi$  can be dependent on  $n$  and  $i$  and a more general class of AR models can be determined." And the footnote referred to the general framework presented in the working paper: Thomas Ho and Sang Bin Lee. "Term Structure Movements and Interest Rates Contingent Claims Pricing." Salomon Brothers Center Series, New York University, 1986.

downstate having equal risk neutral probability of 0.5. A node on the binomial lattice is represented by  $(n,i)$  where  $n$  denotes the time and  $i$  the state, where  $0 \leq n \leq N$  and  $0 \leq i \leq n$  for some large integer  $N$ , the time horizon of the model.

$P(T)$  denotes the price of a zero-coupon bond with a face value of \$1 at time  $T$ , for  $0 \leq T \leq N$ , observed at  $(0,0)$ . This set of prices determines the discount function  $P(\cdot)$  at the initial time. All the agents in the market can observe the discount function. All securities are perfectly divisible and trading has no short selling constraints. These bonds are traded at all the nodes of the binomial lattice, and their prices are denoted by  $P_i^n(T)$  which is the price at state  $i$  at time  $n$  with a remaining maturity of the bond  $T$  periods, and therefore  $P_i^n(\cdot)$  is the discount function at node  $(n,i)$ , and by definition  $P_i^n(0) = 1$  for all  $i$  and  $n$ .

The term structure movement is arbitrage-free if there is no arbitrage opportunity for holding any portfolio of discount bonds at each node point of the binomial lattice over any one period. Such is the case, if the expected “risk neutral” return of holding a discount bond of any maturity  $T$  over any one period is the prevailing one period risk free return. Further, the model value of a  $T$  period discount bond equals the observed bond price  $P(T)$ .

To describe the interest rate model, it suffices to describe the one period discount bond  $P_i^n(1)$ , denoted by  $P_i^n$  for  $0 \leq n \leq N$  and  $0 \leq i \leq n$  for simplicity. This is because, given these one period discount bonds, we can use the backward substitution approach to price any interest rate contingent claim given the cash flows  $C_i^n$  assigned to each node  $(n,i)$ , where  $0 \leq n \leq N$  and  $0 \leq i \leq n$ . Specifically, let  $V_i^n$  be the value of the interest rate contingent claim at time  $n$  and state  $i$ , and then the value of the contingent claim is given by the terminal condition:

$$V_i^N = C_i^N \text{ for } 0 \leq i \leq N \quad (1)$$

and the recursive equation:

$$V_i^n = \frac{1}{2} P_i^n (V_{i+1}^{n+1} + V_i^{n+1}) + C_i^n \text{ for all } 0 \leq n \leq N-1 \text{ and } 0 \leq i \leq n. \quad (2)$$

The value of the contingent claim is given by  $V = V_0^0$ .

### *B. The Model*

The building blocks of the binomial model are the binormal volatilities  $\delta_i^n$ , for  $0 \leq i \leq n$ .  $\delta_i^n$  is the proportional decrease in the one period bond value from state  $i$  to  $i+1$  at time  $n$ . Without loss of generality, we assume that the bond price decreases, and the bond yield

increases, with state  $i$ , and hence  $\delta_i^n < 1$ . When  $\delta_i^n = 1$ , by definition, there is no risk at the binomial node with respect to the upstate and downstate outcomes.

$$\delta_i^n = \frac{P_{i+1}^{n+1}}{P_i^{n+1}} \text{ for all } n, 0 \leq n \leq N-1 \text{ and } 0 \leq i \leq n \quad (3)$$

But there are several requirements imposed on these binomial volatilities.

Condition 1: State dependent volatility requirement asserts that the proportional change of one period bond price is proportional to the interest rate level when the rates are low, constant when it is high.

To specify Condition 1, we begin with the definition of the one period yield:

$$R_i^n = -\log P_i^n / \Delta t, \quad (4)$$

where  $\Delta t$  is the time interval of one period. For example, if one binomial period (the step size of the lattice) is one month, then  $\Delta t$  is 1/12.

According to equation (3), we have

$$\log \delta_i^n = -\log P_i^{n+1} + \log P_{i+1}^{n+1} \quad (5)$$

Substituting equation (4) into (5), we have

$$\log \delta_i^n = (R_i^{n+1} - R_{i+1}^{n+1})\Delta t \quad (6)$$

By the market convention, interest rate volatilities are the standard deviations of proportional change in the annualized yields of the bonds. Let  $\sigma_i^n$  be the annualized volatility at time  $n$  and state  $i$ , noting that the difference of the two binomial outcomes is two standard deviations, then we have:

$$R_{i+1}^{n+1} - R_i^{n+1} = 2\sigma_i^n R_i^n \sqrt{\Delta t} \quad (7)$$

Substitution  $R_{i+1}^{n+1} - R_i^{n+1}$  of equation (7) into (6), and simplify, we derive the relationship of the binomial volatilities and the market convention of interest rate volatilities.

$$\delta_i^n = \exp\left(-2\sigma_i^n R_i^n \Delta t^{3/2}\right) \quad (8)$$

Equation (8) presents the one-to-one relationship between  $\delta_i^n$  and  $\sigma_i^n R_i^n$ . Let  $R$  be some fixed interest rate level, which we call the threshold rate, independent of both  $n$  and  $i$ . We now assume that on the one hand, the interest rate movement is lognormal when  $R_i^n < R$ , and therefore  $\sigma_i^n$  is a function of  $n$ , denoted by  $\sigma(n)$  and not the states  $i$ . On the other

hand, we assume that the interest rate movement becomes normal when  $R_i^n > R$ , evolving from the lognormal process to the normal process continuously. And therefore,  $\sigma_i^n R_i^n$  is independent of the state  $i$ , and equals  $\sigma(n)R$ . This motivates the following specification of  $\sigma_i^n$ ,

$$\sigma_i^n R_i^n = \sigma(n) \min(R_i^n, R) \quad (9)$$

$\sigma(n)$  is some continuous function of time  $n$ , which can be interpreted as the term structure of volatilities<sup>2</sup>. Substitution equation (9) into equation (8), we have the following specification of the forward volatilities:

$$\delta_i^n = \exp\left(-2\sigma(n) \min(R_i^n, R) \Delta t^{3/2}\right) \quad (10)$$

We further assume that the function  $\sigma(n)$  is specified by some parameters  $\sigma_0, \sigma_\infty, \alpha_0, \alpha_1, \alpha_\infty$ :

$$\sigma(n) = (\sigma_0 - \sigma_\infty + \alpha_0 n) \exp(-\alpha_\infty n) + \alpha_1 n + \sigma_\infty \quad (11)$$

The parameters  $\sigma_0, \sigma_\infty, \alpha_0, \alpha_1, \alpha_\infty$  can be interpreted as follow.

$\sigma_0$  is the short rate volatility over the first period. Within the model construction, the short rate is the one period rate.  $\alpha_1 n + \sigma_\infty$  is approximately the short rate forward volatility at time  $n$ , if we assume  $n$  is sufficiently large and the exponential decay term becomes small.  $\alpha_0 + \alpha_1$  and  $\alpha_\infty$  are the short term and long term slopes of the term structure of volatilities. When these parameters are significantly positive, the short term slope would ensure the term structure of volatility would slope upward and the long term slope ensures that the volatility curve decays rapidly.  $\alpha_\infty$  is the proportional decrease of the volatility per year. The steeper is the decay, the stronger is the mean reversion. For this reason, we use the notation  $\alpha$ . This notation is consistent with the literature where  $\alpha$  is often used to denote the speed of mean reversion of the interest rate movements.

The specification of the term structure of volatilities is motivated by the observed market volatility curve. The volatility curve tends to decay exponentially with a term linearly related to time, with a hump in the short to intermediate term at times. Equation (10) and (11) are important in specifying the arbitrage-free interest rate model. Equation (10) ensures interest rates are non-negative and non-explosive, and Equation (11) ensures the mean-reversion behavior.

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<sup>2</sup> Since this is a discrete time model, the interest rates can still become negative as a result of the discrete time approximation, even for some small volatilities when the rates are low. Equation (4) cannot ensure that  $\delta_i^n$  are always bounded by one. For implementation, we use  $\delta_i^n = \exp\left(-2\sigma(n) \max\left(\min(R_i^n, R) \Delta t^{3/2}, \varepsilon\right)\right)$  for some small  $\varepsilon$ , say, 0.0001 or 1 basis point.

## Condition 2. Arbitrage-free condition

The arbitrage-free yield curve movements condition applies to all the bonds with different maturities  $T$ , we therefore need to consider the binomial volatilities with another dimension  $T$ ,  $\delta_i^n(T)$ . Specifically,

$$\delta_i^n(T) = \frac{P_{i+1}^{n+1}(T)}{P_i^{n+1}(T)} \quad 0 \leq T \quad (12)$$

That is, the binomial volatility is the proportion of the  $T$ -year bond at  $(i+1)$ th state to the  $i$ th state at time  $n+1$ . Note that,  $\delta_i^n(0) = 1$ , because the one-period bond has no uncertainties over one period. The volatility for one period bond is  $\delta_i^n(1) = \delta_i^n$ , which are given numbers for the time being.

We will show later that the arbitrage free condition requires that:

$$\delta_i^n(T) = \delta_i^n \delta_i^{n+1}(T-1) \left( \frac{1 + \delta_{i+1}^{n+1}(T-1)}{1 + \delta_i^{n+1}(T-1)} \right) \quad (13)$$

Equation (13) defines the relationships of the binomial volatilities of  $T$ -year bonds, and is important to the construction of the arbitrage-free rate model.

Note that if the binomial volatilities are independent of states, but dependent on time, then the above condition implies that:

$$\delta^n(T) = \delta^n \delta^{n+1} \dots \delta^{n+T-1} \quad (14)$$

When the binomial volatilities are both state and time independent, then

$$\delta^{n+1}(T) = \delta^T, \quad (15)$$

where  $T$  is the power and not a superscript.

Equation (14) and (15) are used in Ho-Lee (2004) and Ho-Lee (1986) respectively. Therefore equation (13) shows that this paper model is a generalization of the previous models.

Given the above conditions, we will also show that the bond pricing model for the one period bond price at node  $(n,i)$  is:

$$P_i^n = \frac{P(n+1)}{P(n)} \prod_{k=1}^n \frac{(1 + \delta_0^{k-1}(n-k))}{(1 + \delta_0^{k-1}(n-k+1))} \prod_{j=0}^{i-1} \delta_j^{n-1} \quad (16)$$

Equation (16) is the specification of the arbitrage-free interest rate model and it is the key result of this paper. Heath, Jarrow, and Morton (HJM) provides an arbitrage-free interest rate model for a given term structure of volatilities, in a continuous time framework. In the same spirit, we here show that as long as the binomial volatilities satisfy equation (13), then the arbitrage-free interest rate model is given by equation (16). Note that, our model requires the recombining of the interest rate movements to maintain the computational tractability, whereas the HJM model does not require the simulated interest rates to recombine. Therefore, our model provides an additional analytical structure. When the binomial step size tends to zero, our discrete time model would converge to a continuous time model. But there is no analogous procedure in converting a continuous model to a recombining binomial lattice model. As a result, an HJM approach does not have a procedure to derive the model presented in this paper.

Equations (13) and (16) specify the interest rate model from the term structure of volatilities. In this paper, we also provide a specific interest rate model, whose term structure of volatilities is given by equations (4), (10) and (11). Thus the set of difference equations (4), (10), (11), (13) and (16) completes the description of the interest rate movement model.

### C. A Recursive Algorithm

Before we show that the equations (13) and (16) ensure the model to satisfy the arbitrage-free condition, we describe a recursive algorithm to generate the one period bond prices  $P_i^n$ .

To construct the arbitrage-free one-period bond prices, we assume that we are given the initial discount function,  $P(T)$ , the threshold rate  $R$ , and the term structure of volatilities  $\sigma(n)$ . The recursive algorithm begins with  $n=0$  and  $i=0$ . According to equation (10), we can solve for  $\delta_0^0$  given  $R_0^0 = -\log P_0^0(1) / \Delta t$ , where  $P_0^0(1) = P(1)$ . Note that equation (13) is automatically satisfied since we assume that  $\delta_i^n(0) = 1$ , as there is no uncertainty initially.

Now, we proceed with the recursive procedure. Let  $n=1$ , equation (16) becomes,

$$P_0^1 = \frac{P(2)(1 + \delta_0^0(0))}{P(1)(1 + \delta_0^0(1))} = \frac{P(2)}{P(1)} \frac{2}{(1 + \delta_0^0)} \quad (17)$$

$$P_1^1 = P_0^1 \delta_0^0 \quad (18)$$

Now, we substitute the values  $P_0^1$  and  $P_1^1$  in equation (4) to derive the forward volatilities for period  $n=1$ .

$$\delta_i^1 = \exp\left(-2\sigma(1) \min(R_i^1, R) \Delta t^{3/2}\right) \text{ for } i = 0, 1. \quad (19)$$

We repeat this procedure for period 2, n=2. Using equation (13), we can now derive the binomial volatilities for term T=2.

$$\delta_i^n(2) = \delta_i^n \delta_i^{n+1} \left( \frac{1 + \delta_{i+1}^{n+1}}{1 + \delta_i^{n+1}} \right) \text{ for } n=0,1 \text{ and } 0 \leq i \leq n. \quad (20)$$

We now use equation (16) to determine the one period bond prices,

$$P_0^2 = \frac{P(3)}{P(2)} \frac{(1 + \delta_0^0)}{(1 + \delta_0^0(2))} \frac{2}{(1 + \delta_0^1)} \quad (21)$$

$$P_1^2 = P_0^2 \delta_0^1 \quad (22)$$

$$P_2^2 = P_0^2 \delta_0^1 \delta_1^1 \quad (23)$$

Using these one period bond prices and equation (10), we can determine the one period binomial volatilities  $\delta_i^2$ , for i= 0, 1, 2.

This completes the specifications for n=2 and we proceed to period 3, n=3. Once again, we begin with equation (13) to determine the forward volatilities of terms greater than 1.

$$\delta_i^n(3) = \delta_i^n \delta_i^{n+1}(2) \left( \frac{1 + \delta_{i+1}^{n+1}(2)}{1 + \delta_i^{n+1}(2)} \right) \quad (24)$$

Then apply equation (16) to determine the one period bond price for state 0.

$$P_0^3 = \frac{P(4)}{P(3)} \frac{(1 + \delta_0^0(2))}{(1 + \delta_0^0(3))} \frac{(1 + \delta_0^1)}{(1 + \delta_0^1(2))} \frac{2}{(1 + \delta_0^2)} \quad (25)$$

We now apply equation (16) repeatedly to determine the one period bond prices for other states.

$$P_1^3 = P_0^3 \delta_0^2, P_2^3 = P_0^3 \delta_0^2 \delta_1^2, P_3^3 = P_0^3 \delta_0^2 \delta_1^2 \delta_2^2 \quad (26)$$

From these one period bond prices, use equation (10) to determine the one-period forward volatilities.

We can describe the recursive procedure more generally. The building blocks of the bond valuation are the binomial volatilities,  $\delta_i^n(T)$ , for n=0,1,..., m; i=0,..., n; T= 0,..., m-n. The recursive procedure begins with a pyramid of the values  $\delta_i^n(T)$  with the top  $\delta_0^0(m)$  and with the base defined by  $\delta_0^{m-1}$ ,  $\delta_{m-1}^{m-1}$ , and  $\delta_0^0$ . We say that the pyramid has dimension

m. Then, the recursive procedure has four steps to increase the dimension of the pyramid from m to m+1:

Step 1: Derivation of the one period bond price for state 0 and time m.

By assumptions of the recursive algorithm, we take the following forward volatilities:

$\delta_0^0(m), \delta_0^1(m-1), \delta_0^2(m-2), \dots, \delta_0^{m-2}(2)$  ,  $\delta_0^0(m-1), \delta_0^1(m-2), \delta_0^2(m-3), \dots, \delta_0^{m-2}(1)$  , and  $\delta_0^{m-1}(1), \delta_1^{m-1}(1), \delta_2^{m-1}(1), \dots, \delta_{m-1}^{m-1}(1)$  as given. Now apply equation (16) to derive the one period discount factor at time m and ith state,  $P_0^m$  .

Step 2: Derivation of the one period discount factors for states 1, ..., m for time m

We use equation (16) repeatedly, to calculate  $P_i^m$  for  $i = 1, \dots, m$  using the one period binomial volatilities,  $\delta_0^{m-1}, \delta_1^{m-1}, \dots, \delta_{m-1}^{m-1}$  . Specifically, we use the following equation,

$$P_i^m = P_0^m \prod_{j=0}^{i-1} \delta_j^{m-1} \text{ for } i = 1, 2, \dots, m \quad (27)$$

Step 3: Derivation of the one period binomial volatilities for time m and states 0, ..., m.

First we use equation (4) to derive  $R_i^m$  from  $P_i^m$  . Then we use equation (10),

$$\delta_i^m(1) = \delta_i^m = \exp\left(-2 \cdot \sigma(m) \min\left(R_i^m, R\right) \Delta t^{3/2}\right) \text{ to calculate } \delta_i^m \text{ for } i = 0, \dots, m$$

This step extends the base of the pyramid of the binomial volatilities by one step, from dimension m-1 to m.

Step 4: Derivation of the higher order of the binomial volatilities

We use Equation (13) to derive an ‘‘additional layer on the face of the pyramid’’. Given  $\delta_i^m(1)$  for  $i = 0, \dots, m$ , we can derive  $\delta_i^{m-1}(2)$  for  $i = 0, \dots, m-1$ , using equation (13). We continue this recursive method until we reach  $\delta_0^0(m+1)$  . This step enables us to add a layer to the face of the pyramid. This layer is defined by starting from the top,  $\delta_0^0(m+1)$  , and then the two binomial volatilities,  $\delta_0^1(m), \delta_1^1(m)$  , and then the three binomial volatilities below,  $\delta_0^2(m-1), \delta_1^2(m-1), \delta_2^2(m-1)$  till we reach the base of the pyramid. This completes the recursive algorithm in that we have increased the dimension of the pyramid from m to m+1.

These four steps of the recursive procedure specify the model using a constructive prescription.

### III. Extensions of the Model

#### A. Determining the T period bond value at node (n,i), $P_i^n(T)$

Thus far we have derived the one period bond prices at any node (n,i). According to equation (16), we can recursively determine the T-period bond prices  $P_i^n(T)$  at any node (n, i) with any maturity T. The bond price is specified by the binomial volatilities,  $\delta_i^n(T)$ , and it is:

$$P_i^n(T) = \frac{P(n+T)}{P(n)} \prod_{k=1}^n \frac{(1 + \delta_0^{k-1}(n-k))}{(1 + \delta_0^{k-1}(n-k+T))} \prod_{j=0}^{i-1} \delta_j^{n-1}(T) \quad (28)$$

$\frac{P(n+T)}{P(n)}$  is the forward price of the T year bond, n period hence, based on the initial

discount function  $P(\cdot)$ .  $\prod_{k=1}^n \frac{(1 + \delta_0^{k-1}(n-k))}{(1 + \delta_0^{k-1}(n-k+T))}$  is the convexity term that adjusts for the convexity of the bonds of maturity T such that all bonds satisfy the arbitrage-free condition whereby the risk neutral expected returns of all bonds equal the one period rate.

$\prod_{j=0}^{i-1} \delta_j^{n-1}(T)$  specifies the proportional change in value of a T-year bond relative to the forward bond price at state i.

Note that, for T=1, equation (28) reduces to the pricing of one-period bond, equation (16).

To illustrate the valuation of a T-period bond, we consider equation (28) for n=2 and 3.

For n= 2, we need to derive first the forward volatilities, a pyramid with dimension T+1.

Then the T period bonds at time n=2 are given by:

$$P_0^2(T) = \frac{P(T+2)}{P(2)} \frac{(1 + \delta_0^0)}{(1 + \delta_0^0(T+1))} \frac{2}{(1 + \delta_0^1(T))} \quad (29)$$

$$P_1^2(T) = P_0^2(T) \delta_0^1(T) \quad (30)$$

$$P_2^2(T) = P_0^2(T) \delta_0^1(T) \delta_1^1(T) \quad (31)$$

And for n=3, we need the pyramid of forward volatilities of dimension T+2, and the bond prices are given by:

$$P_0^3(T) = \frac{P(T+3)}{P(3)} \frac{(1+\delta_0^0(2))}{(1+\delta_0^0(T+2))} \frac{(1+\delta_0^1)}{(1+\delta_0^1(T+1))} \frac{2}{(1+\delta_0^2(T))} \quad (32)$$

$$\begin{aligned} P_1^3(T) &= P_0^3(T)\delta_0^2(T), \\ P_2^3(T) &= P_0^3(T)\delta_0^2(T)\delta_1^2(T), \\ P_3^3(T) &= P_0^3(T)\delta_0^2(T)\delta_1^2(T)\delta_2^2(T) \end{aligned} \quad (33)$$

### B. The $m$ -factor model

The one-factor model can be extended to two-factor model in a straightforward manner. We assume that there are two independent factors. Each factor is specified by a binomial lattice. The two lattices are combined into a multinomial model, with 0.25 risk neutral probability for each of the four outcomes for each step.

The intuitive idea is based on the early observation that the yield of a bond is the sum of three parts: the forward rate, the convexity adjustment, and the stochastic movement. A two-factor model is the combined movement of two independent factors in  $i$  and  $j$  directions. If we project the movement in one direction and focus only on the  $i$  movements, then we should have a factor arbitrage-free movement. Similarly, if we focus on the  $j$  direction, the movement is also arbitrage-free. The co-movement is the sum of the two arbitrage-free movements. Given this intuitive motivation, the two factor arbitrage-free model is given below. Let  $P_{i,j}^n(T)$  be the price of a  $T$  year bond at time  $n$ , at state  $(i, j)$ . Then the bond price is specified by combining two one-factor models. Specifically, we have

$$P_{i,j}^n(T) = \frac{P(n+T)}{P(n)} \prod_{k=1}^n \frac{(1+\delta_{0,1}^{k-1}(n-k))}{(1+\delta_{0,1}^{k-1}(n-k+T))} \frac{(1+\delta_{0,2}^{k-1}(n-k))}{(1+\delta_{0,2}^{k-1}(n-k+T))} \prod_{k=0}^{i-1} \delta_{k,1}^{n-1}(T) \prod_{k=0}^{j-1} \delta_{k,2}^{n-1}(T) \quad (34)$$

where

$$\begin{aligned} \delta_{i,1}^n(T) &= \delta_{i,1}^n \delta_{i,1}^{n+1}(T-1) \left( \frac{1+\delta_{i+1,1}^{n+1}(T-1)}{1+\delta_{i,1}^{n+1}(T-1)} \right), \\ \delta_{i,2}^n(T) &= \delta_{i,2}^n \delta_{i,2}^{n+1}(T-1) \left( \frac{1+\delta_{i+1,2}^{n+1}(T-1)}{1+\delta_{i,2}^{n+1}(T-1)} \right). \end{aligned} \quad (35a)$$

and the one period forward volatilities are given by

$$\delta_{i,1}^m(1) = \delta_{i,1}^m \text{ by definition of } \delta_{i,1}^m,$$

and by extending the specification of  $\delta_i^m$  to the two factor model, we have,

$$\delta_{i,1}^m = \exp\left(-2 \cdot \sigma_1(m) \min(R_{i,1}^m, R) \Delta t^{3/2}\right).$$

Similarly, we can define  $\delta_{i,2}^m$  for the other factor, and we have

$$\delta_{i,2}^m(1) = \delta_{i,2}^m = \exp\left(-2 \cdot \sigma_2(m) \min(R_{i,2}^m, R) \Delta t^{3/2}\right). \quad (35b)$$

Using the direct extension, we can specify the one period rates for the two factor model for any future period m and state i, and  $R_{i,1}^m$  and  $R_{i,2}^m$  are defined by

$$R_{i,1}^m \Delta t = -\log\left(\frac{P(n+1)}{P(n)}\right) + \sum_{k=0}^{n-1} \log\left(\frac{(1 + \delta_{0,1}^{k-1}(n-k))}{(1 + \delta_{0,1}^{k-1}(n-k+T))}\right) + \sum_{j=0}^{i-1} \delta_{k,1}^{n-1}(T),$$

and

$$R_{i,2}^m \Delta t = -\log\left(\frac{P(n+1)}{P(n)}\right) + \sum_{k=0}^{n-1} \log\left(\frac{(1 + \delta_{0,2}^{k-1}(n-k))}{(1 + \delta_{0,2}^{k-1}(n-k+T))}\right) + \sum_{j=0}^{i-1} \delta_{k,2}^{n-1}(T). \quad (35c)$$

For clarity of exposition, we have presented the two-factor model. The generalization of the two-factor model to the multifactor model is clear. For an m-factor model, the bond price is represented by  $P_{i_1 i_2 \dots i_m}^n(T)$ .

The model shows that the price of a T period bond at any state and time n has three components, as described above. First,  $\frac{P(n+T)}{P(n)}$  represents the forward price based on the initial yield curve. Second, C(n,T) is the convexity term given by:

$$C(n, T) = \prod_{k=1}^n \frac{(1 + \delta_{0,1}^{k-1}(n-k)) \dots (1 + \delta_{0,m}^{k-1}(n-k))}{(1 + \delta_{0,1}^{k-1}(n-k+T)) \dots (1 + \delta_{0,m}^{k-1}(n-k+T))}. \quad (36)$$

This term adjusts the bond price such that the additional returns derived from the convexity of the bond are countered by this adjustment to assure the arbitrage-free condition. Third, the stochastic movement term  $\Lambda(i_1, \dots, i_m)$  is given by

$$\Lambda(i_1, \dots, i_m) = \prod_{k=0}^{i_1-1} \delta_{k,1}^{n-1}(T) \dots \prod_{k=0}^{i_m-1} \delta_{k,m}^{n-1}(T). \quad (37)$$

Then the m-factor bond price is given by:

$$P_{i_1 i_2 \dots i_m}^n(T) = \frac{P(n+T)}{P(n)} C(n, T) \Lambda(i_1, \dots, i_m). \quad (38)$$

### C. The arbitrage-free interest rate movement condition

We now need to show that the model equation (10) satisfies the two conditions of arbitrage-free interest rate movements. Specifically, we require that:

Condition 1. Consistency with the one period discount factor

$$P_{i_1 i_2 \dots i_m}^n(T) = 2^{-m} P_{i_1 i_2 \dots i_m}^n \cdot \sum_{k_1=0}^1 \dots \sum_{k_m=0}^1 P_{i_1+k_1, \dots, i_m+k_m}^{n+1}(T-1) \quad (39)$$

This condition requires all bonds to have a risk neutral expected return over one period at each node to be the risk free rate, a condition consistent with Harris and Kreps (1979).

Condition 2: Consistency with the discount function

If  $P_{i_1 i_2 \dots i_m}^n(0) = 1$  for a given  $n$ , and for all  $0 \leq i_1, \dots, i_m \leq n$ , equation (11) would imply that  $P_{0, \dots, 0}^0(n) = P(n)$ , the given  $n$ -period initial discount bond price. The following proposition provides the proof.

Proposition 1. The Model for Bond Prices

We consider a two-factor model in this proof, and the results can be generalized to higher dimension. Let  $P(T)$  be the initial discount function. The price of a  $T$  period zero coupon bond with \$1 principal at time  $n$  and state  $i$  and  $j$  is given by equation (38):

$$P_{i,j}^n(T) = \frac{P(n+T)}{P(n)} \prod_{k=1}^n \frac{(1 + \delta_{0,1}^{k-1}(n-k))}{(1 + \delta_{0,1}^{k-1}(n-k+T))} \frac{(1 + \delta_{0,2}^{k-1}(n-k))}{(1 + \delta_{0,2}^{k-1}(n-k+T))} \prod_{k=0}^{i-1} \delta_{k,1}^{n-1}(T) \prod_{k=0}^{j-1} \delta_{k,2}^{n-1}(T) \quad (40)$$

where the binomial volatilities are related by equation (35)

$$\begin{aligned} \delta_{i,1}^n(T) &= \delta_{i,1}^n \delta_{i,1}^{n+1}(T-1) \left( \frac{1 + \delta_{i+1,1}^{n+1}(T-1)}{1 + \delta_{i,1}^{n+1}(T-1)} \right), \\ \delta_{i,2}^n(T) &= \delta_{i,2}^n \delta_{i,2}^{n+1}(T-1) \left( \frac{1 + \delta_{i+1,2}^{n+1}(T-1)}{1 + \delta_{i,2}^{n+1}(T-1)} \right), \end{aligned}$$

where  $\delta_{i,1}^n(0) = 1$ ,  $\delta_{i,2}^n(0) = 1$ ,  $\delta_{i,1}^n(1) = \delta_{i,1}^n$  and  $\delta_{i,2}^n(1) = \delta_{i,2}^n$  which are given. (41)

The T-year bond price  $P_{i,j}^n(T)$  satisfies the arbitrage-free condition of equation (39)

$$P_{i,j}^n(T+1) = 2^{-2} P_{i,j}^n \sum_{i=0}^1 \sum_{j=0}^1 P_{i,j}^{n+1}(T) \quad (42)$$

Proof:

We begin with the one factor model in the proof. According to the pricing model, equation (28), we have:

$$P_i^n(T) = \frac{P(n+T)}{P(n)} \frac{(1 + \delta_0^0(n-1))}{(1 + \delta_0^0(n-1+T))} \frac{(1 + \delta_0^1(n-2))}{(1 + \delta_0^1(n-2+T))} \cdots \frac{(1 + \delta_0^{n-1}(0))}{(1 + \delta_0^{n-1}(T))} \delta_0^{n-1}(T) \dots \delta_{i-1}^{n-1}(T) \quad (43)$$

We need to show that the arbitrage-free condition holds, i.e:

$$P_i^n(T) = \frac{1}{2} P_i^n (P_i^{n+1}(T-1) + P_{i+1}^{n+1}(T-1)) \quad (44)$$

Now, we can express

$$P_i^n = \frac{P(n+1)}{P(n)} \frac{(1 + \delta_0^0(n-1))}{(1 + \delta_0^0(n))} \frac{(1 + \delta_0^1(n-2))}{(1 + \delta_0^1(n-1))} \cdots \frac{(1 + \delta_0^{n-1}(0))}{(1 + \delta_0^{n-1}(1))} \delta_0^{n-1} \dots \delta_{i-1}^{n-1} \quad (45)$$

$$P_i^{n+1}(T-1) = \frac{P(n+T)}{P(n+1)} \frac{(1 + \delta_0^0(n))}{(1 + \delta_0^0(n-1+T))} \frac{(1 + \delta_0^1(n-1))}{(1 + \delta_0^1(n-2+T))} \cdots \frac{(1 + \delta_0^n(0))}{(1 + \delta_0^n(T-1))} \delta_0^n(T-1) \dots \delta_{i-1}^n(T-1) \quad (46)$$

$$P_{i+1}^{n+1}(T-1) = P_i^{n+1}(T-1) \delta_i^n(T-1) \quad (47)$$

By a direct computation, using equation (43), (45), (46) and (47), equation (44) can be derived. We note that by assumption, equation (41) for the one factor model, we have:

$$\frac{\delta_k^n(T)}{\delta_k^{n+1}(T-1) \delta_k^n} = \frac{(1 + \delta_{k+1}^{n+1}(T-1))}{(1 + \delta_k^{n+1}(T-1))} \text{ for } k=0, \dots, i. \quad (48)$$

Similarly, we can show that the arbitrage-free condition holds for higher dimension.

To illustrate, we first consider  $n=1$ , we have from equation (40) with  $i=j=0$ , we have:

$$P_{0,0}^1(T) = \frac{P(T+1)}{P(1)} \frac{2}{(1+\delta_{0,1}^0(T))} \frac{2}{(1+\delta_{0,2}^0(T))}. \quad (49)$$

Now we need to show that given equation (49), the arbitrage free condition, letting  $n=0$  in Equation (11), below holds:

$$P_{0,0}^0(T+1) = 2^{-2} P_{0,0}^0 \sum_{i=0}^1 \sum_{j=0}^1 P_{i,j}^1(T). \quad (50)$$

Note that  $P_0^0(1) = P(1)$  and  $P_0^0(T) = P(T)$ , and use equation (40), the right hand side of equation (50) becomes:

$$\frac{1}{4} P(1) \cdot \frac{P(T+1)}{P(1)} \frac{2}{(1+\delta_{0,1}^0(T))} \frac{2}{(1+\delta_{0,2}^0(T))} \left[ 1 + \delta_{0,1}^0(T) + \delta_{0,2}^0(T) + \delta_{0,1}^0(T) \cdot \delta_{0,2}^0(T) \right]. \quad (51)$$

In simplifying the above expression, it is reduced to  $P(T)$  and hence, we have shown that the right hand side equals the left hand side of equation (50).

Next, we consider the case  $n=2$ . The arbitrage-free condition, Equation (50), becomes:

$$P_{0,0}^1(T+1) = 2^{-2} P_{0,0}^1 \sum_{i=0}^1 \sum_{j=0}^1 P_{i,j}^2(T). \quad (52)$$

By using equation (40) for the specific node points and maturity, we have the following equations (53) to (58).

$$P_{0,0}^1(T+1) = \frac{P(T+2)}{P(1)} \frac{2}{(1+\delta_{0,1}^0(T+1))} \frac{2}{(1+\delta_{0,2}^0(T+1))} \quad (53)$$

$$P_{0,0}^1 = \frac{P(2)}{P(1)} \frac{2}{(1+\delta_{0,1}^0)} \frac{2}{(1+\delta_{0,2}^0)} \quad (54)$$

$$P_{0,0}^2(T) = \frac{P(2+T)}{P(2)} \frac{(1+\delta_{0,1}^0)}{(1+\delta_{0,1}^0(1+T))} \frac{2}{(1+\delta_{0,1}^1(T))} \frac{(1+\delta_{0,2}^0)}{(1+\delta_{0,2}^0(1+T))} \frac{2}{(1+\delta_{0,2}^1(T))} \quad (55)$$

$$P_{1,0}^2(T) = P_{0,0}^2(T) \delta_{0,1}^1(T), \quad (56)$$

$$P_{0,1}^2(T) = P_{0,0}^2(T) \delta_{0,2}^1(T) \quad (57)$$

$$P_{1,1}^2(T) = P_{0,0}^2(T) \delta_{0,1}^1(T) \delta_{0,2}^1(T) \quad (58)$$

Substituting these equations to equation (52) and simplify, we show that the arbitrage-free condition of equation (52) is satisfied.

A more general proof is given in Appendix A.

Q.E.D.

Corollary: The model prices of the zero coupon bonds are consistent with the initial observed discount function  $P(T)$ .

Proof: In the special base, for  $n = i_1 = i_2 = \dots = i_m = 0$ , according to the equation (10),

$$P_{0,0,\dots,0}^0(T) = P(T)$$

Q.E.D.

## IV. The Model Calibration Results

### A. Data

We now apply the model specified by equations (4), (10), (11), (13) and (16) to the observed swaption prices in the market. Garban International, a broker in the swaption market, provided the data. The sample period is from June 30, 2003 till June 30, 2004, using monthly data. The swaptions expire in 1, 2, 3, 4, 5, 7, and 10 years. The option gives the holder the right to enter into a swap at a specified fixed rate with tenor 1, 2, ..., 10 years. The fixed rates are set such that the swaptions are at-the-money at the time of pricing. The prices of the swaptions are quoted in volatilities ( %) based on the Black model for swaptions. There are 70 observations for each date. The swaption prices for June 30, 2004 are presented in Appendix B for illustration. The key interest rates, the swap spot rates, of that date are given below for reference.

Years	0.25	0.5	1	2	3	4	5	6	7	8	9	10	15	20
Yields	1.61%	1.94%	2.47%	3.15%	3.68%	4.13%	4.38%	4.64%	4.85%	5.03%	5.04%	5.14%	5.82%	6.05%

### B. Calibration Procedure

For each date, we determine the term structure of volatilities such that the model prices fit the observed swaption prices. To calibrate the one-factor model, we use a non-linear search procedure in determining the parameters of the function  $\sigma(n)$ , which are  $\sigma_0, \sigma_\infty, \alpha_0, \alpha_1, \alpha_\infty$  according to equation (11), such that the sum of squared proportional errors between the observed and model prices of the benchmark securities is minimized.

Specifically,  $P_j^o, P_j^t$  be the observed and theoretical swaption prices respectively, for  $j = 1 \dots 70$ . We define the objective function be:

$$J(\sigma_0, \sigma_1, \sigma_\infty, \alpha_0, \alpha_\infty) = \sum_{j=1}^{70} \left( \frac{P_j^o - P_j^t}{P_j^o} \right)^2 \quad (59)$$

We then use a non-linear optimization procedure to determine the minimized objective function  $J$ , denoted by  $J^*$  by changing the choice variables  $\sigma_0, \sigma_\infty, \alpha_0, \alpha_1, \alpha_\infty$ . For the two-factor model, we would have six choice variables,  $\sigma_0, \sigma_\infty, \alpha_0, \alpha_1, \alpha_\infty$  for the first factor and a constant  $\sigma$  for the second factor for equation (35b). The mean error of the estimation, given that there are 70 observed prices for each date, is defined as

$$e = \sqrt{J^*} / 70 \quad (60)$$

We assume that threshold rate  $R$  of equation (10) to be 3%. We also consider the alternative model assuming  $R$  to be 9%. We have tested the model with different threshold rates, and we have found that the model performs best with 3%. For this reason, we use the 3% model as a benchmark.

The binomial lattice model is based on a one month step size. To value a 10 year swaption with the option expiring in 10 years, our lattice needs to incorporate a 20 year term structure, or 240 monthly steps. Therefore, we use a yield curve extending to 20 years, based on the swap rate.

### *C. The Empirical Model and the Davidson and Mackinnon C Test*

To test the explanatory power of the two-factor model, we use the Davidson and Mackinnon C test. Let  $P_j^o$  be the  $j$ th swaption observed price,  $j = 1, \dots, 70$ , on a particular date. Let  $\Phi_i(X_i, \beta)$  be the proposed model, where  $X_i$  is a vector of observations which are the spot rates and the swaption prices;  $\beta$  is a vector of parameters to be estimated and  $\varepsilon_{0i}$  is an error term  $N(0, \theta)$ . Let  $\Gamma_i(Z_i, \gamma)$  be the alternative model, where  $Z_i$  is a vector of observations on the exogenous variables;  $\gamma$  is a vector of parameters to be estimated; and  $\varepsilon_{1i}$  is an error term  $N(0, \theta^*)$ .

Let the null hypothesis to be:

$$H_0 : P_j^o = \Phi_j(X_j, \beta) + P_j^o \varepsilon_{0j} \quad (61)$$

and the alternative hypothesis is

$$H_1 : P_j^o = \Gamma_j(Z_j, \gamma) + P_j^o \varepsilon_{1i} . \quad (62)$$

The C test assumes that the observed swaption prices can be explained by a convex combination of the two models, as specified below:

$$P_j^o = (1 - \alpha)\Phi_j + \alpha\Gamma_j + P_j^o \varepsilon_j \quad (63)$$

where  $\varepsilon_j$  is assumed to be normal with zero mean.

Rearranging, we get

$$(P_j^o - \Phi_j)/P_j^o = \alpha((P_j^o - \Phi_j)/P_j^o - (P_j^o - \Gamma_j)/P_j^o) + \varepsilon_j \quad (64)$$

Equation (64) says that if the alternative model  $\Gamma_j$  can provide better explanatory power to the model  $\Phi_j$  (in terms of minimizing the sum of squared proportional errors), then  $\alpha$  should be significantly different from 0. This hypothesis can be tested using a conventional asymptotic t-test.

#### D. The Two-Factor Model versus the One-Factor Model

We compare the estimation errors of the one factor model, equation (13) and (16) and the alternative two-factor model, based on equations (35) and (36). In both cases, we use 3% for the threshold rate  $R$ . The mean percentage errors of the models are reported in column A and B in Table I. We then apply the C test (equation (64)) to the two models for each date, and the coefficients and the t-statistics are reported in Column C, D, and E.

**Table I**  
**A Comparison of the One-Factor and the Two-Factor Models**

	1&2 factor model of mean errors		C test		
	A	B	C	D	E
dates	2-factor model*	1-factor model*	Coefficients $\alpha$	t-statistics	R-square(%)
06/30/03	3.51%	4.34%	1.969	7.636	46
07/30/03	3.73%	4.42%	2.776	8.119	49
08/29/03	2.28%	2.93%	3.048	13.19	72
09/30/03	3.49%	4.28%	3.215	11.02	64
10/30/03	2.60%	3.34%	2.973	12.95	71
11/28/03	1.50%	2.20%	1.981	13.08	71
12/31/03	1.83%	2.65%	2.623	19.92	85
01/31/04	1.72%	2.49%	2.42	16.52	80
02/27/04	1.65%	2.33%	2.626	16.96	81
03/31/04	2.22%	3.11%	2.944	21.6	87
05/31/04	1.03%	1.44%	1.989	11.34	65
06/30/04	2.09%	2.14%	1.877	2.062	6
Average	2.30%	2.97%	2.537	12.866	65

\* Note that the threshold rate is 3%.

Based on the mean errors reported in Table I, the empirical results support the validity of both the two-factor and one-factor models. The models have reasonably small errors in the calibration. The results are particularly reasonable when we consider that the two-factor model and the one-factor model use only six and five parameters respectively for the calibration. Further, the models are calibrating to some long dated swaptions where the swaption market is less liquid. The price quotes are subject to the bid-ask spread of the market and quotes are dependent on the size of the transaction. These considerations can result in a price quoted in a range of 2%.

The t-statistics (column D) and the adjusted R squared (column E) of the C test support the hypothesis that the two-factor model provides higher explanatory power than the one-factor model for all the dates, with R squared exceeding 80% for some dates. The average percentage errors over the sample period for the two-factor model and the one factor model are 2.30% and 2.97% respectively showing that the two-factor model provides a lower average error.

The model provides further insight into the explanatory power of the two factor model. Equation (64) can be re-arranged as follows.

$$\frac{(P_j^o - \Gamma_j)}{P_j^o} = \frac{(\alpha - 1)}{\alpha} \frac{(P_j^o - \Phi_j)}{P_j^o} + \frac{1}{\alpha} \varepsilon_j.$$

Since the coefficients  $\alpha$  are estimated between approximately 2.0 and 3.2, the empirical model shows that the 2-factor model percentage error for each swaption is between 0.5 and 0.69 of that of the 1-factor model plus a random error term. The errors may be small as suggested by the high t-statistics and the R-squared values, depending on the sample date.

#### *E. The Normal Model versus the Lognormal Model*

The Ho-Lee and Hull-White models assume that the market perceives the interest rate movements to be a normal model. By way of contrast, the Black, Derman, and Toy and Black-Karasinski models assume that the interest rate movements to be approximated by a lognormal model. Which assumption is more empirically correct based on the sample swaption prices? We use our two-factor model to provide some insights into this question. Specifically, we compare the model using a threshold rate of 3% to the alternative model using a higher threshold rate of 9%. The former model would support the interest rate movement be better explained by a normal model, while the latter would support a lognormal interest rate movement. The results are reported below, in Table II. Column A and Column B provides the mean error of each date for the two factor model with the threshold rates 3% and 9% respectively. Column C, D and E provide the coefficients, test statistics, and the adjusted R squared for the C test respectively.

**Table II**  
**A Comparison of a Normal-like Model and a Lognormal-like Model**

Dates	2 factor model mean errors		C test		
	A	B	C	D	E
	3% threshold rate	9% threshold rate	Coefficients $\alpha$	t-statistics	R-square(%)
06/30/03	3.51%	3.79%	1.289	3.494	15
07/30/03	3.73%	4.08%	2.558	4.888	26
08/29/03	2.28%	2.67%	2.542	7.136	42
09/30/03	3.49%	3.84%	2.129	4.74	25
10/30/03	2.60%	2.97%	1.184	4.697	24
11/28/03	1.50%	1.86%	1.067	6.132	35
12/31/03	1.83%	2.25%	1.919	7.348	44
01/31/04	1.72%	2.04%	1.733	6.15	35
02/27/04	1.65%	1.89%	1.88	5.507	31
03/31/04	2.22%	2.62%	2.048	6.172	36
05/31/04	1.03%	1.36%	1.576	8.259	50
06/30/04	2.09%	1.48%	0.0635	0.5661	0
Average	2.30%	2.57%	1.666	5.424	30

The results show that the model with a lower threshold rate (3%) has generally lower mean errors than that of a higher threshold rate (9%), except, for the date June 30, 2004. The difference is also shown to be significant by the t-statistics. Therefore the empirical results suggest that the model is more akin to the normal model. Note that R can be a choice parameter in the calibration procedure. Our research has shown that in general, the optimized threshold rate is around 3% for our sample period.

## V. Model Results

### *A. The Yields of the Zero Coupon Bonds*

We here determine the yield of a zero coupon bond. For clarity of the exposition, we will present the results for the one factor model. The extension of the one-factor model to the

m-factor model is straightforward. The yields are then used to depict the interest rate movements, based on the calibrated interest rate model.

By definition, the yield of a T-period bond at node (n,i) is given by

$$R_i^n(T) = -\frac{1}{T\Delta t} \log P_i^n(T). \quad (65)$$

Using equation (65) and equation (34), we can derive the yield of the T year bond at each node (n,i). It is given below:

$$R_i^n(T)T\Delta t = -\log \frac{P(n+T)}{P(n)} - \sum_{k=1}^n \log \frac{(1+\delta_0^{k-1}(n-k))}{(1+\delta_0^{k-1}(n-k+T))} - \sum_{j=0}^{i-1} \log \delta_j^{n-1}(T) \quad (66)$$

In particular, the yield for the one period bond is:

$$R_i^1\Delta t = -\log \frac{P(n+1)}{P(n)} - \sum_{k=1}^n \log \left( \frac{(1+\delta_0^k(n-k))}{(1+\delta_0^k(n-k+1))} \right) - \sum_{j=0}^{i-1} \log \delta_j^{n-1} \quad (67)$$

For illustration, the one period yield for n=1 are:

$$R_0^1\Delta t = -\log \frac{P(2)}{P(1)} - \log \frac{2}{(1+\delta_0^0)} \quad (68)$$

$$R_1^1\Delta t = -\log \frac{P(2)}{P(1)} - \log \frac{2}{(1+\delta_0^0)} - \log \delta_0^0 \quad (69)$$

The model is used to calculate the interest rates at each node point of the lattices. They are depicted below.

### B. Analysis of the Interest Rate Movements

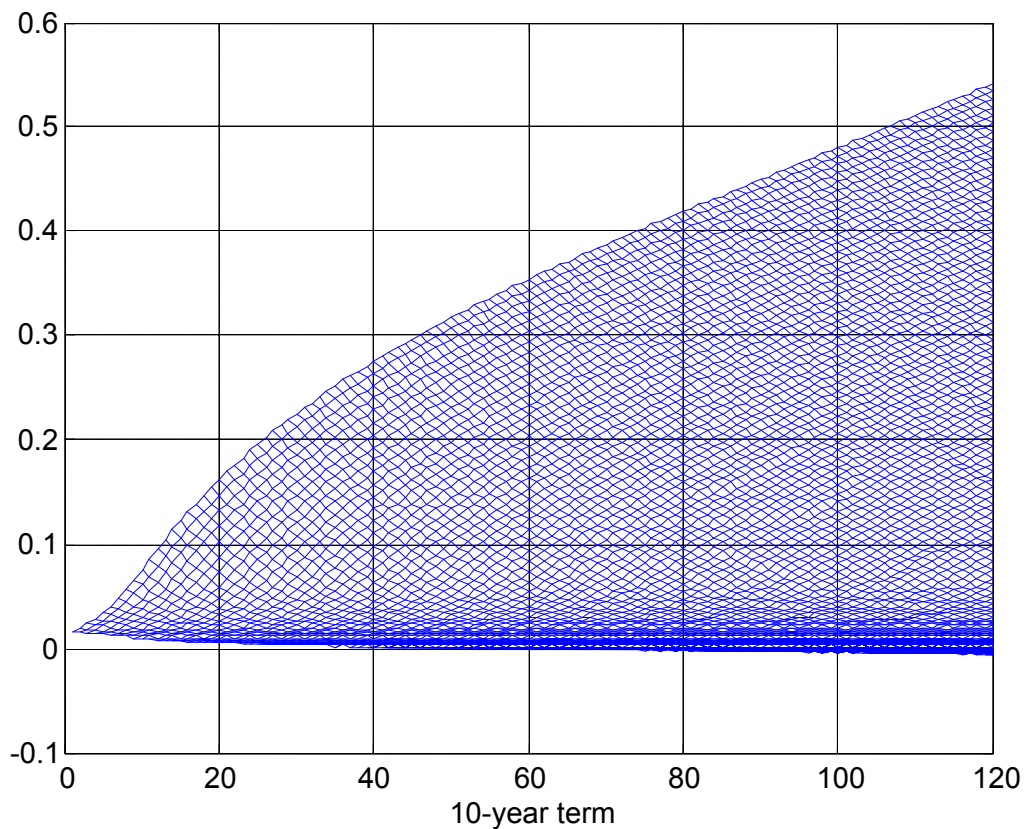
Given the above empirical results, we now proceed to study the simulated yield curve movements based on the two-factor model with 3% threshold rate using equation (69). We will provide three main observations: (a) The simulated interest rates exhibit mean reversion, no negative and unreasonable high interest rates. (b) The simulated instantaneous principal movements of the yield curve are consistent with the historical movements. (c) The simulated yield curves can be inverted.

#### B.1. Interest Rates Distribution

Figure 1 depicts the one factor lattice model. It shows that the model has almost no negative interest rates and minimal extremely high interest rates. There are some negative interest rates with negligible absolute value because of the discrete time approximation of the lognormal process. (Footnote 2 provides the explanation of negative rates more

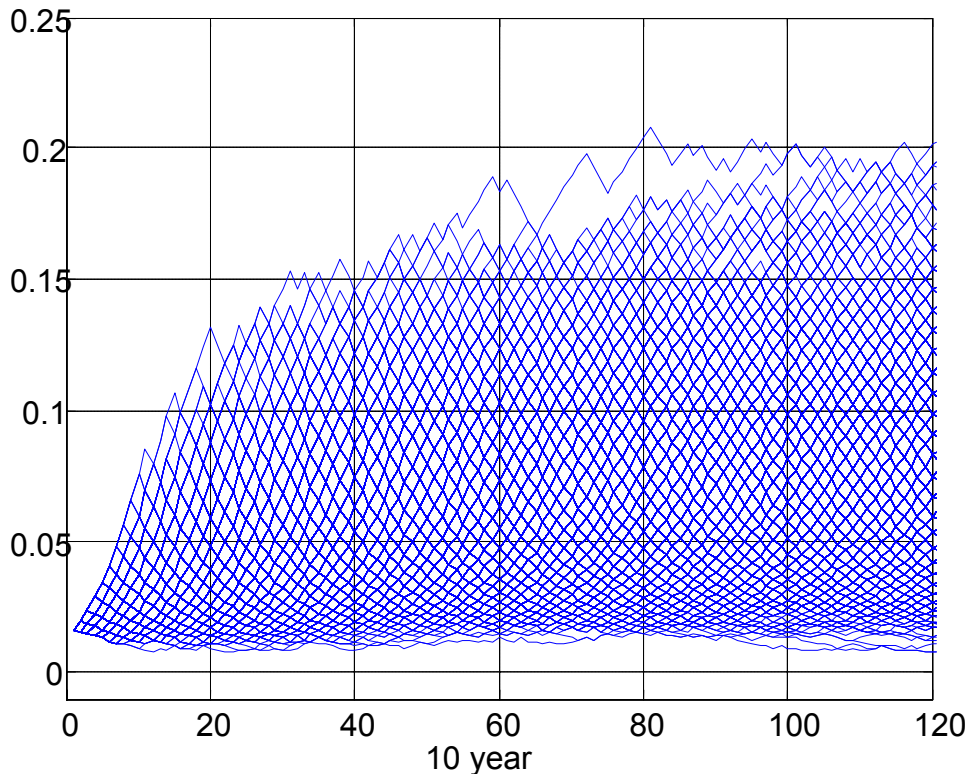
precisely). Also note that the lattice tends not to expand with time indicating the mean reversion behavior of the interest rate model.

**Figure 1. One Month Interest Rate on One Factor Model**



We should also note that the extremely high and low interest rates are unlikely outcomes based on risk neutral probabilities. When we simulate the scenario paths in the lattices, most of the paths in fact appear in the middle part of the lattice. This result is demonstrated in Figure 2. The results of randomly simulating 1,000 scenarios from the two factor model show that all paths of the one month rates are bounded from zero and most paths lie between 0 and 10% of the interest rate levels.

**Figure 2. Randomly Simulated Paths of the 2 –Factor Model**



*B.2. Term Structure of Volatilities and Co-Movements of the Yield Curve*

We now consider the term structure of volatilities of the two-factor model. The estimated function of the steepening movement  $\sigma(n)$  of equation (11) for each date is presented in Table III below.

**Table III**  
**The Estimated Volatilities of the Steepening Movement Factor**  
**(June 30, 03 ~ June 30, 04)**

Year	0.25	0.5	1	2	3	4	5	6	7	8	9	10	15
6/30/03	56.1%	55.2%	53.4%	49.9%	46.5%	43.1%	39.9%	36.7%	33.7%	30.7%	27.8%	24.9%	11.8%
7/30/03	38.4%	40.7%	44.1%	47.1%	46.8%	44.7%	41.7%	38.4%	35.1%	31.9%	28.9%	26.2%	15.3%
8/29/03	65.7%	63.5%	59.5%	53.0%	47.7%	43.4%	39.7%	36.4%	33.4%	30.5%	27.9%	25.3%	12.9%
9/30/03	72.9%	70.0%	64.7%	56.0%	49.3%	44.0%	39.7%	36.0%	32.8%	29.8%	27.1%	24.5%	11.8%
10/30/03	59.8%	58.7%	56.5%	52.3%	48.3%	44.5%	40.8%	37.3%	34.0%	30.8%	27.7%	24.7%	11.4%
11/28/03	47.7%	48.8%	50.1%	50.0%	47.9%	44.9%	41.5%	38.0%	34.7%	31.5%	28.6%	25.8%	13.9%
12/31/03	61.6%	59.1%	55.1%	49.3%	45.2%	41.8%	38.7%	35.7%	32.8%	29.9%	27.0%	24.0%	9.5%
1/30/04	62.8%	60.0%	55.3%	48.8%	44.4%	40.9%	37.8%	35.0%	32.3%	29.6%	26.9%	24.3%	11.2%
2/27/04	60.4%	59.0%	56.3%	51.7%	47.8%	44.5%	41.5%	38.9%	36.4%	34.0%	31.8%	29.7%	19.5%
3/31/04	63.2%	61.2%	57.5%	51.5%	46.6%	42.5%	38.9%	35.6%	32.5%	29.6%	26.8%	24.0%	10.5%
5/31/04	51.6%	50.9%	49.4%	46.6%	43.9%	41.4%	38.9%	36.6%	34.3%	32.2%	30.1%	28.1%	18.8%
6/30/04	42.1%	42.6%	43.0%	42.0%	40.1%	38.0%	36.1%	34.4%	32.9%	31.7%	30.6%	29.6%	25.5%

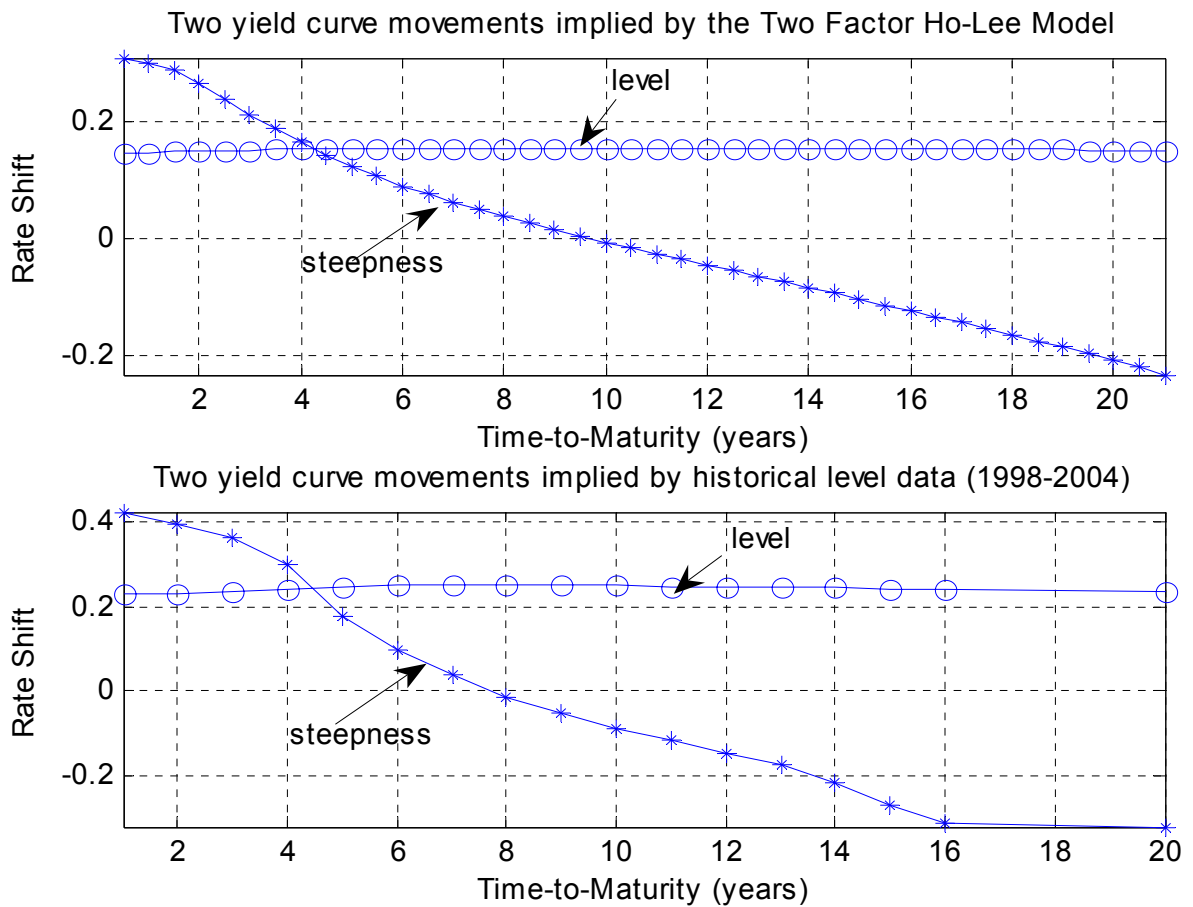
The results show that the steepening movement requires the function to decline monotonically, except for 7/30/03 and 11/28/03, in this sample period, inducing mean reversion in the process. This is consistent with the observed market term structure of volatilities where it may rise in the short term segment of the curve.

Next, we use the model to construct the “instantaneous” variance-covariance of the interest rates. Specifically, for each date, we calibrate the model to the observed swaption prices. Then, based on the model, we consider the four possible spot yield curves, at the nodes one step from the initial point. The model assumes that the risk neutral probabilities are 0.25 for each outcome, we can calculate the variance-covariance matrix of the interest rates for terms 1, 2, ..., 10 years, based on the proportional change in the yields of the bonds. Using this matrix we can determine the principal components of the yield curve movements. These principal components are called the “implied yield curve movements.” Since these implied yield curve movements are based on a two-factor model calibrated from the observed swaption prices, we only consider the first two movements.

The results show that the implied principal components maintain a similar shape over the sample period. A representative steepening movement based on June 30, 2004 is presented on the upper panel of Figure 3.

We use the historical monthly data from February 1998 till August 2004 of the yield curve. Using the proportional changes of the yield curve over the one month period, we determine the variance-covariance matrix of the interest rates. The steepening movement of the principal components is depicted in the lower panel of Figure 3. The upper and lower panels of Figure 3 show that the implied steepening movement is similar to the historical steepening movement.

**Figure 3. Two Yield Curve Movements Implied by the Two Factor Model**



The results show that the implied principal movements of the yield curve, based on the observed swaption prices, are similar to those estimated from the historical data. Furthermore, the percentages of the movement explained by the parallel movements over the period of our observation are also consistent with that estimated from historical data.

Table IV presents the percentage of the yield curve movements explained by the implied parallel movement of the yield curve over the sample period.

**Table IV**  
**Percentage of the Movements Explained by the Parallel Shifts of the Yield Curve**  
**According to the Model**

6/30/03	7/30/03	8/29/03	9/30/03	10/30/03	11/28/03	12/31/03	1/31/04	2/27/04	3/31/04	5/31/04	6/30/04
99.08%	99.51%	98.99%	98.78%	99.02%	99.32%	98.86%	98.93%	99.56%	98.88%	99.66%	99.99%

### *B.3. Inverted Yield Curve*

While the above results suggest that the parallel movement of the yield curve is dominant, the simulated yield curves do exhibit inversion, with the short rate level higher than the long rate level, due to the steepening yield curve movements.

We consider a five year (60 month ) time horizon. Based on the 60\*60 possible outcomes, the 15 year rates range from 1.47% to 23.44%. With the steepening movement, the two year rates vary over a larger range of 1.13% to 26.11%. Indeed, if we consider the spread, defined as the 15 year rate net of the 2 year rate, with a positive value to mean the yield curve inverted, then we can see that inverted yield curves are exhibited at many nodes.

**Table V**  
**The Spread between the Long Rate and the Short Rate (15 year rate – 2 year rate)**

	j=60	54	48	42	36	30	24	18	12	6	0
i=60	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267
54	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198
48	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129
42	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061
36	0.00084	0.00084	0.00084	0.00084	0.00084	0.00084	0.0009	0.0008	0.0009	0.0009	0.0008
30	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078
24	0.0149	0.0148	0.0149	0.0149	0.0149	0.0149	0.0149	0.0149	0.0148	0.0149	0.0149
18	0.0136	0.0136	0.0136	0.0135	0.0136	0.01361	0.01354	0.01357	0.0136	0.01358	0.01358
12	0.0078	0.0078	0.0079	0.00789	0.00782	0.00785	0.00785	0.00785	0.00785	0.00788	0.00785
6	0.0045	0.0044	0.0044	0.00446	0.00443	0.00443	0.00443	0.00443	0.00443	0.00439	0.0045
0	0.0034	0.0034	0.00346	0.00345	0.00345	0.00345	0.00345	0.00345	0.00341	0.0035	0.0034

The results show that the yield curve is inverted when the state i equals or exceeds 42.

## **VI. Implications of the Model**

The model has a broad range of applications. The model can be used in trading, portfolio management, hedging, measuring key rate durations, convexities and other parametrics

for measuring risks. The model can also meet the urgent needs of the financial markets in recent years.

Today, the regulators demand more financial risk disclosure; auditors require more detail understanding of internal financial models, and risk managers need fair valuation of balance sheets items, in addition to the tradable securities. New regulations like International Financial Reporting Standard 39 requiring fair value accounting; Sarbanes-Oxley's operational risk regulation in the documentation of business processes; Financial Accounting Standard 123 requiring the fair value of employee options as expense, reporting of Value-at-Risk measures are some of the examples that prompted the new challenges in the specifications the interest rate model that is a key component of many valuation and risk models.

For these reasons, financial engineers, auditors and regulators prefer the interest rate models to have the following properties: (1) The interest rate scenarios should be consistent with historical experiences, like, exhibiting non-negative interest rates, possible inverted yield curve, and mean reversion of interest rate level. (2) An ability to value a broad range of securities and balance sheet items accurately and the model can be calibrated to a broad range of marketable securities, not just swaptions or caps/floors, and can provide all the discount function and the term structure of volatilities at each node. (3) The model is transparent and the model results can be duplicated quite readily. Our multifactor interest rate model, with the forward volatilities being both state and time dependent can address all these requirements. We now discuss each requirement in turn.

When the interest rate model can generate the scenarios consistent to the historical experience, risk managers can relate the fair valuation of financial instruments with the stress scenarios. In managing the downside risk of a position, like using the Value-at-Risk measures, risk managers often need to better understand the distributions of the interest rates in the extreme cases as simulated by the valuation model. The proposed model can provide the realistic scenarios even in the extreme cases. The proposed model enables the risk managers to study the pathwise values of a portfolio of interest rate contingent claims based on a certain confidence level, where this model can provide reasonable extreme scenarios. (The pathwise value is defined as the present value of the cashflows along an interest rate path.)

In enterprise risk management and in satisfying regulations, the interest rate model should be able to value a broad range of financial instruments in a consistent fashion. Otherwise, we would have to resort to using different interest rate models for different interest rate contingent claims. Since the proposed model can value many benchmark securities accurately without using Monte-Carlo simulation, the model can be calibrated to the market using a broad range of market benchmark securities, which may be a portfolio of European, American, and Asian options. This is particularly useful in managing the balance sheet items, which can be complex. Some of the balance sheet items are more akin to American or Asian options, and therefore, it would be more appropriate to value them relative to the American and Asian benchmark derivatives. This model can also effectively value securities like mortgages where the prepayments at each node on the lattice would also depend on the yield curve shape and the term structure of volatilities. Using the proposed model, we can determine these prepayments

in a consistent fashion since the prevailing yield curve and volatility curve are both consistent with the arbitrage-free condition.

Transparency is an important attribute. The Black-Scholes model, the Black models for caps/floors and swaptions have such an attribute. They can provide a benchmark for comparing models and market standards in measuring the market implied values, like the Black implied volatilities. The proposed model can be accurately specified by only the parameters  $\sigma, \sigma_0, \sigma_\infty, \alpha_0, \alpha_1, \alpha_\infty$  that specify the term structure of volatilities (or just specifying the term structure of volatilities like quoting the spot yield curve), and  $R$  the threshold interest rate. As long as the users of the model agree on these parameters, they should derive the same model results, and the model can be used for a broad range of contingent claims as discussed above. Any model that requires Monte-Carlo simulations like the Brace, Gatarek, and Musiela (Jackel and Rebonato (2000)) or the string models, or the construction of the lattice requiring numerical procedures in pruning the tree like Hull-White, or numerical methods in building an interest rate tree like Black, Derman, and Toy would not be able to offer a transparent specification of the model.

## VII. Conclusions

This paper presents an interest rate model that has four desirable attributes: (1) An arbitrage-free model that takes the yield curve as given. (2) A recombining lattice model for accurate price of American and some Asian options. (3) A multifactor model to capture the yield curve shape movements. (4) A state-time dependent term structure of volatilities that ensures mean-reversion, non-negative interest rates and not unacceptably high interest rate levels.

We have also presented empirical evidence in supporting the validity of the model. Further, we have shown that the simulated yield curves based on the model are consistent with the behavior of the yield curve historically.

The model is specified by a set of difference equations, and therefore it is simple to implement. It is also simple to be verified. For these reasons, the model can have broad applications, including meeting some of the regulatory requirements in the use of financial models in financial reporting.

In research literature, the Ho-Lee model (1986) is often cited as simply an arbitrage-free model, a model that can be equivalently expressed in a continuous time framework. We show that a recombining binomial lattice model has more analytical structure that provides efficiency in computation. For example, we can use backward substitution methodology to value options. Further, the model specifies the discount function and term structure of volatilities at each node, such that, at any node point, these discount functions in turn satisfy all arbitrage-free conditions. Such analytical structure not only offers a broader range of applications, it can be exploited for other applications. These extensions will be left for future research.

## Appendix A : Proof of Proposition 1

The discount function with the time to maturity T at time n and state i

$$P_i^n(T) = \frac{P(n+T)}{P(n)} \prod_{k=1}^n \frac{(1 + \delta_0^{k-1}(n-k))}{(1 + \delta_0^{k-1}(n-k+T))} \prod_{j=0}^{i-1} \delta_j^{n-1}(T) \quad (\text{A.1})$$

The discount function with the time to maturity 1 at time n and state i

$$P_i^n(1) = \frac{P(n+1)}{P(n)} \prod_{k=1}^n \frac{(1 + \delta_0^{k-1}(n-k))}{(1 + \delta_0^{k-1}(n-k+1))} \prod_{j=0}^{i-1} \delta_j^{n-1}(1) \quad (\text{A.2})$$

Divide Equation (A.1) by Equation (A.2) to get  $\frac{P_i^n(n+1)}{P_i^n(1)}$

$$\frac{P_i^n(n+1)}{P_i^n(1)} = \frac{P(n+T)}{P(n+1)} \prod_{k=1}^n \frac{(1 + \delta_0^{k-1}(n-k+1))}{(1 + \delta_0^{k-1}(n-k+T))} \frac{\prod_{j=0}^{i-1} \delta_j^{n-1}(T)}{\prod_{j=0}^{i-1} \delta_j^{n-1}(1)} \quad (\text{A.3})$$

The discount function with the time to maturity T-1 at time n+1 and state i

$$P_i^{n+1}(T-1) = \frac{P(n+1+T-1)}{P(n+1)} \prod_{k=1}^{n+1} \frac{(1 + \delta_0^{k-1}(n+1-k))}{(1 + \delta_0^{k-1}(n+1-k+T-1))} \prod_{j=0}^{i-1} \delta_j^n(T-1) \quad (\text{A.4})$$

$$= \frac{P(n+1+T-1)}{P(n+1)} \prod_{k=1}^n \frac{(1 + \delta_0^{k-1}(n+1-k))}{(1 + \delta_0^{k-1}(n+1-k+T-1))} \cdot \frac{(1 + \delta_0^n(n+1-n-1))}{(1 + \delta_0^n(n+1-n-1+T-1))} \prod_{j=0}^{i-1} \delta_j^n(T-1)$$

$$= \frac{P(n+T)}{P(n+1)} \prod_{k=1}^n \frac{(1 + \delta_0^{k-1}(n+1-k))}{(1 + \delta_0^{k-1}(n-k+T))} \cdot \frac{(1 + \delta_0^n(0))}{(1 + \delta_0^n(T-1))} \prod_{j=0}^{i-1} \delta_j^n(T-1)$$

$$= \frac{P(n+T)}{P(n+1)} \prod_{k=1}^n \frac{(1 + \delta_0^{k-1}(n+1-k))}{(1 + \delta_0^{k-1}(n-k+T))} \cdot \frac{2}{(1 + \delta_0^n(T-1))} \prod_{j=0}^{i-1} \frac{\delta_j^{n-1}(T)}{\delta_j^{n-1}} \frac{(1 + \delta_j^n(T-1))}{(1 + \delta_{j+1}^n(T))}$$

$$\because \delta_j^{n-1}(T) = \delta_j^{n-1} \delta_j^n(T-1) \frac{(1 + \delta_{j+1}^n(T-1))}{(1 + \delta_j^n(T-1))} \quad \text{from Equation (6)}$$

$$= \frac{P(n+T)}{P(n+1)} \prod_{k=1}^n \frac{(1 + \delta_0^{k-1}(n+1-k))}{(1 + \delta_0^{k-1}(n-k+T))} \prod_{j=0}^{i-1} \frac{\delta_j^{n-1}(T)}{\delta_j^{n-1}} \cdot \frac{2}{(1 + \delta_i^n(T-1))} \quad (\text{A.5})$$

Comparing Equation (A.5) with Equation (A.3), we see that

$$P_i^{n+1}(T-1) = \frac{P_i^n(T)}{P_i^n(1)} \frac{2}{(1 + \delta_i^n(T-1))} \quad (\text{A.6})$$

Similarly, we can show that

$$P_{i+1}^{n+1}(T-1) = \frac{P_i^n(T)}{P_i^n(1)} \frac{2 \delta_i^n(T-1)}{(1 + \delta_i^n(T-1))} \quad (\text{A.7})$$

Plugging Equation (A.6) and Equation (A.7) into Equation (44), we can see that Equation (44) holds.

$$P_i^n(T) = \frac{1}{2} P_i^n(P_i^{n+1}(T-1) + P_{i+1}^{n+1}(T-1)) \quad (44)$$

as required.

For the two factor model, we can show that

$$P_{i+1,j+1}^{n+1}(T-1) = \frac{P_{i,j}^n(T)}{P_{i,j}^n(1)} \frac{2 \delta_{i,1}^n(T-1)}{(1 + \delta_{i,1}^n(T-1))} \frac{2 \delta_{j,2}^n(T-1)}{(1 + \delta_{j,2}^n(T-1))} \quad (\text{A.8})$$

$$P_{i+1,j}^{n+1}(T-1) = \frac{P_{i,j}^n(T)}{P_{i,j}^n(1)} \frac{2 \delta_{i,1}^n(T-1)}{(1 + \delta_{i,1}^n(T-1))} \frac{2}{(1 + \delta_{j,2}^n(T-1))} \quad (\text{A.9})$$

$$P_{i,j+1}^{n+1}(T-1) = \frac{P_{i,j}^n(T)}{P_{i,j}^n(1)} \frac{2}{(1 + \delta_{i,1}^n(T-1))} \frac{2 \delta_{j,2}^n(T-1)}{(1 + \delta_{j,2}^n(T-1))} \quad (\text{A.10})$$

$$P_{i,j}^{n+1}(T-1) = \frac{P_{i,j}^n(T)}{P_{i,j}^n(1)} \frac{2}{(1 + \delta_{i,1}^n(T-1))} \frac{2}{(1 + \delta_{j,2}^n(T-1))} \quad (\text{A.11})$$

From Equation (A.8), (A.9), (A.10) and (A.11), we can see that Equation (A.12) holds.

$$P_{i,j}^n(T) = \frac{1}{4} P_{i,j}^n(P_{i+1,j+1}^{n+1}(T-1) + P_{i+1,j}^{n+1}(T-1) + P_{i,j+1}^{n+1}(T-1) + P_{i,j}^{n+1}(T-1)) \quad (\text{A.12})$$

We can show that the m factor model satisfies the no-arbitrage condition in the same way.

## Appendix B : Swaption Black Volatilities on June 30, 2004

Option Term (years)	Swap Tenor(years)									
	1yr	2yr	3yr	4yr	5yr	6yr	7yr	8yr	9yr	10yr
1yr	30.95	28.31	25.99	24.24	22.65	21.54	20.83	20.00	19.20	18.55
2yr	24.94	23.37	21.95	21.18	20.05	19.27	18.87	18.30	17.70	17.16
3yr	22.15	20.74	20.06	18.97	18.25	17.67	17.16	16.90	16.50	16.07
4yr	20.38	19.41	18.36	17.67	17.30	16.83	16.38	15.90	15.60	15.18
5yr	18.78	18.03	17.34	16.73	16.35	15.92	15.51	15.10	14.80	14.43
7yr	16.50	15.64	15.13	14.85	14.35	14.04	13.47	13.50	13.20	12.73
10yr	13.47	12.97	12.56	12.16	11.77	11.81	11.60	11.40	11.20	10.99

## Appendix C : Percentage Errors of the Swaption Estimated Prices

<b>Swaption Error (% Diff) June 30, 04</b>					
Option Term	Swap Tenor	Two factor beta9	Two factor beta3	One factor beta3	
1	1	(0.85)	(1.80)	(2.57)	
2	1	1.15	0.99	0.66	
3	1	1.80	2.84	2.95	
4	1	1.05	1.59	2.07	
5	1	0.25	1.01	1.76	
7	1	0.28	1.19	1.94	
10	1	1.44	2.48	3.21	
1	2	(0.08)	(0.13)	0.63	
2	2	(0.62)	(1.13)	(0.38)	
3	2	(0.31)	(0.94)	(0.22)	
4	2	(1.79)	(2.98)	(3.38)	
5	2	(0.29)	0.35	0.20	
7	2	(1.39)	(1.65)	(1.70)	
10	2	0.31	0.39	0.39	
1	3	(0.70)	(0.29)	(0.28)	
2	3	(0.61)	0.09	0.10	
3	3	(0.59)	(1.12)	(1.08)	
4	3	(0.41)	(1.37)	(1.33)	
5	3	(0.24)	(1.21)	(1.19)	
7	3	0.03	(0.92)	(0.93)	
10	3	1.07	3.41	3.03	
1	4	(1.24)	(1.18)	(1.41)	
2	4	0.28	0.73	0.58	
3	4	(1.24)	(0.37)	(0.48)	
4	4	(1.37)	(0.13)	(0.23)	
5	4	(2.76)	(3.09)	(3.17)	
7	4	(2.56)	(3.37)	(3.46)	
10	4	(0.67)	(1.42)	(1.54)	
1	5	0.19	(0.45)	(0.60)	
2	5	0.77	0.18	(0.00)	
3	5	1.33	(1.48)	(1.62)	
4	5	1.55	0.76	0.66	
5	5	(0.12)	0.11	0.03	
7	5	(0.40)	0.54	0.46	
10	5	(0.89)	(1.87)	(1.92)	
1	6	(0.56)	(1.96)	(2.04)	
2	6	0.09	(1.07)	(1.19)	
3	6	0.37	(0.57)	(0.72)	
4	6	1.68	0.88	0.70	
5	6	2.16	1.41	1.20	
7	6	2.61	3.97	3.89	
10	6	1.81	3.81	3.75	
1	7	1.16	3.71	3.65	
2	7	(1.09)	(1.27)	(1.31)	
3	7	(0.30)	(1.08)	(1.14)	
4	7	0.40	(0.12)	(0.20)	
5	7	1.05	0.78	0.66	
7	7	1.61	1.49	1.35	
10	7	2.75	2.74	2.58	
1	8	3.36	3.38	3.20	
2	8	1.80	5.98	5.98	
3	8	(2.93)	(5.89)	(5.86)	
4	8	(2.44)	(5.60)	(5.60)	
5	8	(0.59)	(2.54)	(2.55)	
7	8	(0.61)	(1.72)	(1.74)	
10	8	0.52	(0.03)	(0.05)	
1	9	(0.30)	(0.53)	(0.55)	
2	9	3.15	3.05	3.06	
3	9	2.86	1.93	1.97	
4	9	1.40	(0.44)	(0.37)	
5	9	(1.53)	1.31	1.45	
7	9	(2.37)	0.98	1.18	
10	9	(2.44)	1.11	1.39	
1	10	(2.46)	1.14	1.38	
2	10	(2.52)	1.06	1.19	
3	10	(0.56)	1.34	1.40	
4	10	(0.32)	0.05	0.05	
5	10	0.10	(0.89)	(0.95)	
7	10	0.57	(1.68)	(1.78)	
10	10	0.96	(2.45)	(2.59)	

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